Problem-1

The series is a convergent geometric series so that

$$| e^c | = e^c < 1 \text{ thus we must have } c < 0$$

$$\sum_{n=0}^{+\infty} e^{nc} = \frac{1}{1-e^c} = D \Rightarrow 1 - e^c = \frac{1}{D} \Rightarrow 1 - \frac{1}{D} = e^c$$

$$c = \ln\left(1 - \frac{1}{D}\right) < 0 \text{ as it should because D} > 1.$$

Problem 2

If $a_n = \dfrac{(-3)^n x^n}{n^{3/2}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{(-3)^n x^n}\right| = \lim_{n\to\infty}\left|-3x\left(\frac{n}{n+1}\right)^{3/2}\right| = 3\,|x| \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^{3/2}$$

$$= 3\,|x|\,(1) = 3\,|x|$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \dfrac{(-3)^n}{n\sqrt{n}} x^n$ converges when $3\,|x| < 1 \quad \Leftrightarrow \quad |x| < \frac{1}{3}$, so $R = \frac{1}{3}$. When $x = \frac{1}{3}$, the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^{3/2}}$ converges by the Alternating Series Test. When $x = -\frac{1}{3}$, the series $\sum_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ is a convergent p-series $\left(p = \frac{3}{2} > 1\right)$. Thus, the interval of convergence is $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

Problem-3

$$f(x)=\frac{A}{x-B}=\frac{A}{(x-c)-(B-c)}=-\frac{A}{B-c}\underline{\frac{1}{1-[(x-c)/(B-c)]}}$$

underlined fraction is a geometric series which converges for $\left|\frac{x-c}{B-c}\right|<1$

Thus

$$f(x)=-\frac{A}{B-c}\sum_{n=0}^{\infty}\left(\frac{x-c}{B-c}\right)^n=\sum_{n=0}^{\infty}\left(-\frac{A}{(B-c)^{n+1}}\right)(x-c)^n$$

Interval of convergence : $\mathrm{I}=(c-|B-c|, \mathrm{c}+|\mathrm{B}-\mathrm{c}|)$

Problem-4

The auxiliary equation is $ar^2+br+c=0$.

If $b^2-4ac>0$, then any solution is of the form

$y(x)=c_1e^{r_1x}+c_2e^{r_2x}$ where $r_1=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

But a , b , and c are all positive so both r_1 and r_2 are negative and $y(x)=0$

If $b^2-4ac=0$, then any solution is of the form

$y(x)=c_1e^{rx}+c_2xe^{rx}$ where $r=-b/(2a)<0$ since a , b are positive. Hence $y(x)=0$.

if $b^2-4ac<0$
then any solution is of the form $y(x)=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$ where $\alpha=-b/(2a)<0$ since a and b are positive. Thus $y(x)=0$.

problem 5

After substitution of $W(x)$ we obtain

$$\sum_{n=-\infty}^{+\infty} y_n \left(j\frac{n\pi}{L}\right)^m e^{j\frac{n\pi x}{L}} + A \sum_{n=-\infty}^{+\infty} y_n \, e^{j\frac{n\pi x}{L}} = \sum_{n=-\infty}^{+\infty} f_n e^{j\frac{n\pi x}{L}}$$

$$\to \sum_{n=-\infty}^{+\infty} \left\{ y_n \left[\left(j\frac{n\pi}{L}\right)^m + A \right] - f_n \right\} e^{j\frac{n\pi x}{L}} = 0$$

$$\to \quad y_n \left[\left(j\frac{n\pi}{L}\right)^m + A \right] - f_n = 0$$

$$\Rightarrow \quad y_n = \frac{f_n}{\left(j\frac{n\pi}{L}\right)^m + A}$$

Therefore we have

$$\underbrace{W(x) = \sum_{n=-\infty}^{+\infty} \frac{f_n}{\left(j\frac{n\pi}{L}\right)^2 + A} e^{j\frac{n\pi x}{L}}}_{\text{solution in complex form}}$$

Problem 6

a)

$$\cos^2(2\pi k_0 x) = \frac{1}{4}\left\{ e^{j2\pi(2k_0)x} + e^{-j2\pi(2k_0)x} + 2 \right\}$$

$$\int_{-\infty}^{+\infty} \cos^2(2\pi k_0 x) e^{-j2\pi k x} dx =$$

$$= \frac{1}{4} \underbrace{\int_{-\infty}^{+\infty} e^{-j2\pi(k-2k_0)x} dx}_{\delta(k-2k_0)} + \frac{1}{4} \underbrace{\int_{-\infty}^{+\infty} e^{-j2\pi(k+2k_0)x} dx}_{\delta(k+2k_0)} + \frac{1}{2} \underbrace{\int_{-\infty}^{+\infty} e^{-j2\pi k x} dx}_{\delta(k)}$$

Thus we have

$$F(k) = \frac{1}{4}\left\{ \delta(k-2k_0) + \delta(k+2k_0) + 2\,\delta(k) \right\}$$

b)

$$\cos^3(2\pi k_0 x) = \frac{1}{8}\left\{ e^{j2\pi(3k_0)x} + e^{-j2\pi(3k_0)x} + 3e^{j2\pi k_0 x} + 3e^{-j2\pi k_0 x} \right\}$$

$$\int_{-\infty}^{\infty} \cos^3(2\pi k_0 x)\, e^{-j2\pi k x} dx =$$

$$= \frac{1}{8}\left\{ \underbrace{\int_{-\infty}^{+\infty} e^{-j2\pi(k-3k_0)x} dx}_{\delta(k-3k_0)} + \underbrace{\int_{-\infty}^{+\infty} e^{-j2\pi(k+3k_0)x} dx}_{\delta(k+3k_0)} + 3\underbrace{\int_{-\infty}^{+\infty} e^{-j2\pi(k-k_0)x} dx}_{\delta(k-k_0)} + 3\underbrace{\int_{-\infty}^{+\infty} e^{-j2\pi(k+k_0)x} dx}_{\delta(k+k_0)} \right\}$$

Thus

$$F(k) = \frac{1}{8}\left\{ \delta(k-3k_0) + \delta(k+3k_0) + 3\,\delta(k-k_0) + 3\,\delta(k+k_0) \right\}$$